

# TVID: Three-loop Vacuum Integrals from Dispersion relations

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## Abstract

TVID is a program for the numerical evaluation of general three-loop vacuum integrals with arbitrary masses. It consists of two parts. An algebraic module, implemented in Mathematica, performs the separation of the divergent pieces of the master integrals and identifies special cases. The numerical module, implemented in C, carries out the numerical integration of the finite pieces. In this note, the structure of the program is explained and a few usage examples are given.

# 1 Overview

**Program name and version:** TVID, version 1.0 (December 2016).

**System requirements:** LINUX-compatible platform; GNU C compiler GCC 4.4 or similar; MATHEMATICA 10.X.

**Copyright:** The TVID source code may be freely used and incorporated into other projects, but the authors ask that always a reference to this document and to Ref. [1] be included.

**External code elements included:** TVID uses the Gauss-Kronrod routine QAG from the QUADPACK library [2], translated into C++, and the C++ package DOUBLEDDOUBLE for 30 digit floating point arithmetic [3].

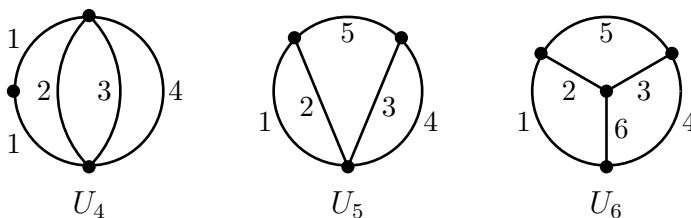
**Code availability:** The TVID source code is available for download at <http://www.pitt.edu/~afreitas/>.

## 2 Introduction

TVID is a public computer code for the efficient and robust numerical evaluation of a generic basis of master integrals for the three-loop vacuum diagrams. It is based on an approach using dispersion relations, which is described in Ref. [1]. The method makes use of ideas which were previously developed for the numerical evaluation of two-loop self-energy integrals [4].

Analytical results for three-loop vacuum integrals are known for cases with one or two independent mass scales [5–8]. However, for fully general mass configurations, only numerical approaches are currently available. Besides the technique used in TVID, an alternative numerical method based on differential equations has been discussed in Ref. [9].

TVID provides results for the divergent and finite pieces of the three-loop vacuum master integrals. For some applications it may be necessary to evaluate the master integrals to higher orders in  $\epsilon$ , which requires a non-trivial extension of the method of Ref. [1].



**Figure 1:** Basic master integral topologies used by TVID. The dot indicates a propagator that is raised to the power 2.

The set of master integrals is shown in Fig. 1 and defined by

$$U_4(m_1^2, m_2^2, m_3^2, m_4^2) \equiv M(2, 1, 1, 1, 0, 0), \quad (1)$$

$$U_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \equiv M(1, 1, 1, 1, 1, 0), \quad (2)$$

$$U_6(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \equiv M(1, 1, 1, 1, 1, 1), \quad (3)$$

where

$$\begin{aligned} M(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \\ = i \frac{e^{3\gamma_E \epsilon}}{\pi^{3D/2}} \int d^D q_1 d^D q_2 d^D q_3 \frac{1}{[q_1^2 - m_1^2]^{\nu_1} [(q_1 - q_2)^2 - m_2^2]^{\nu_2}} \\ \times \frac{1}{[(q_2 - q_3)^2 - m_3^2]^{\nu_3} [q_3^2 - m_4^2]^{\nu_4} [q_2^2 - m_5^2]^{\nu_5} [(q_1 - q_3)^2 - m_6^2]^{\nu_6}}, \quad (4) \end{aligned}$$

and  $\epsilon = (4 - D)/2$ ,  $D$  is the number of dimensions in dimensional regularization. For generic values of the mass parameters, any three-loop vacuum integral can be reduced to a linear combination of  $U_4$ ,  $U_5$  and  $U_6$  functions with the help of integration-by-parts identities [10].

The master integrals  $U_4$ ,  $U_5$  and  $U_6$  have the following symmetry properties:

- $U_4(m_1^2, m_2^2, m_3^2, m_4^2)$  is invariant under arbitrary permutations of  $m_{2,3,4}$ .
- $U_5(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$  is invariant under the replacements  $\{m_1 \leftrightarrow m_2\}$ ,  $\{m_3 \leftrightarrow m_4\}$ , and  $\{m_1 \leftrightarrow m_3, m_2 \leftrightarrow m_4\}$ , as well as any combination thereof.
- $U_6(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2)$  is invariant under the replacements  $\{m_2 \leftrightarrow m_3, m_1 \leftrightarrow m_4\}$ ,  $\{m_2 \leftrightarrow m_6, m_4 \leftrightarrow m_5\}$ ,  $\{m_1 \leftrightarrow m_6, m_3 \leftrightarrow m_5\}$ , and any combination thereof.

### 3 Structure and usage of the program

#### 3.1 Numerical part

The numerical part of TVID is programmed in C and evaluates the finite remainder functions defined in Ref. [1]. These are

$$\begin{aligned} U_{4,\text{sub}}(m_1^2, m_2^2, m_3^2, m_4^2) &= U_4(m_1^2, m_2^2, m_3^2, m_4^2) - U_4(m_1^2, 0, m_3^2, 0) - U_4(m_1^2, 0, 0, m_4^2) \\ &\quad + 2 U_4(m_1^2, 0, 0, 0) \end{aligned} \quad (5)$$

for  $m_1 \neq 0$ , see section 3.1 in Ref. [1];

$$\begin{aligned} U_{4,\text{sub},0}(m_2^2, m_3^2, m_4^2) \\ &= U_4(0, m_2^2, m_3^2, m_4^2) - \left[ B_0(0, 0, 0) - B_0(0, \delta, \delta) - \log \frac{\delta^2}{m_2^2} \right] T_3(m_2^2, m_3^2, m_4^2) \\ &\quad - \log \frac{\delta^2}{m_2^2} \sum_{i=2}^4 T_3(m_i^2, 0, 0) \end{aligned} \quad (6)$$

$$= U_{4,\text{sub}}(\delta^2, m_2^2, m_3^2, m_4^2) + \log \frac{\delta^2}{m_2^2} \left[ T_3(m_2^2, m_3^2, m_4^2) - \sum_{i=2}^4 T_3(m_i^2, 0, 0) \right], \quad (7)$$

where  $\delta > 0$  is an infinitesimally small parameter, see section 3.2 in Ref. [1];

$$U_{5,\text{sub}}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) = M(2, 1, 1, 2, 1, 0; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) \quad (8)$$

for the finite part of the generic  $U_5$  case, where  $M$  is defined in eq. (4), see section 4.1 in Ref. [1];

$$\begin{aligned} U_{5,\text{sub},0}(m_3^2, m_4^2, m_5^2) &= M(1', 1, 1, 2, 1, 0; m_4^2, 0, m_3^2, m_4^2, m_5^2) \\ &= i \frac{e^{3\gamma_E \epsilon}}{\pi^{3D/2}} \int d^D q_1 d^D q_2 d^D q_3 \frac{1}{q_1^2 [q_1^2 - m_4^2] (q_1 - q_2)^2} \\ &\quad \times \frac{1}{[(q_2 - q_3)^2 - m_3^2] [q_3^2 - m_4^2]^2 [q_2^2 - m_5^2]}, \end{aligned} \quad (9)$$

for the special case  $U_5(0, 0, m_3^2, m_4^2, m_5^2)$ , see section 4.3 in Ref. [1]; and

$$\begin{aligned} U_{6,\text{sub}}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) &= U_6(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \\ &\quad - U_6(m_6^2, m_6^2, m_6^2, m_6^2, m_6^2, m_6^2), \end{aligned} \quad (10)$$

where, without loss of generality, it has been assumed that  $m_6 \geq m_i$  for  $i = 1, \dots, 5$ , see section 5 in Ref. [1].

The numerical code is called with the command

```
ucall infile outfile
```

where *infile* is the name of the input file, and *outfile* is the name of the file where the results shall be placed. *infile* may contain a list of lines, separated by line breaks, where each line has the form

```
fname massA massB ...
```

Here *fname* is the name of the function to be evaluated, see Tab. 1, and *massA*, *massB*, etc. are the numerical mass parameters supplied. For example,

```
U4 1 2 3 4
U50 1.5 2.5 0.5
```

asks for the evaluation of  $U_{4,\text{sub}}(1, 2, 3, 4)$  and of  $U_{5,\text{sub},0}(1.5, 2.5, 0.5)$ . When `ucall` is completed, it fills *outfile* with a list of the numerical results, again separated by line breaks. For instance, the example above will return

```
-5.555128856244808e1
-1.537493378796251
```

Internally, the numerical code uses the Gauss-Kronrod routine QAG from the QUADPACK library [2] to evaluate the dispersion integrals. This routine has been translated into C++ from the original FORTRAN code, and amended to facilitate 30 digit floating point arithmetic from the package DOUBLEDDOUBLE [3].

Function	Mass parameters	Symbol <i>fname</i> used by numerical code <code>ucall</code>	Symbol used by algebraic Mathematica code
$U_{4,\text{sub}}$	$m_1, m_2, m_3, m_4$	U4	U4sub
$U_{4,\text{sub},0}$	$m_2, m_3, m_4$	U40	U4sub0
$U_{5,\text{sub}}$	$m_1, m_2, m_3, m_4, m_5$	U5	M21121
$U_{5,\text{sub},0}$	$m_3, m_4, m_5$	U50	M1p1121
$U_{6,\text{sub}}$	$m_1, m_2, m_3, m_4, m_5, m_6$	U6	U6sub

**Table 1:** Symbols for basic finite remainder functions used in the numerical and algebraic parts of TVID.

### 3.2 Algebraic part

The algebraic part of TVID runs in Mathematica 10 [11]. It performs the separation of divergent and finite pieces of the master integrals and identifies cases that require special treatment. The program is loaded into a Mathematica session with

```
<< i3.m
```

It contains two basic functions, `PrepInt` and `UCall`.

`PrepInt` takes any  $U_4$ ,  $U_5$  or  $U_6$  function as input and splits them into divergent terms and the finite remainder functions introduced in the previous subsection. It returns a series expansion in  $\epsilon$ . For example

```
In[2]:= PrepInt[U4[1.,2.,2.,1.]]
```

```
Out[2]= 1.66667 2.61371 7.48687
----- + ----- + ----- +
      3      2      $eps
     $eps     $eps
```

```
> ((22.6599 - 2.36848 10-15 I) + U4sub[1., 1., 2., 2.]) + 0[$eps]
```

Here `U4sub` is the function  $U_{4,\text{sub}}$  introduced in eq. (5). The list of all symbols returned by `PrepInt` is given in the last column of Tab. 1.

`UCall` invokes the numerical code `ucall` (see previous subsection) to evaluate the finite remainder functions in the output of `PrepInt`. For the example above this leads to

```
In[3]:= UCall[%]
```

```
Out[3]= 1.66667 2.61371 7.48687 -15
----- + ----- + ----- + (-2.36076 - 2.36848 10-15 I) + 0[$eps]
      3      2      $eps
```

Technically, the executable `ucall` is called through an external operating system command, using the Mathematica function `Run`. The Mathematica function `UCall` looks for the executable `ucall` in the directory given by the variable `$Directory`, which by default is set to `"/"`. For passing input and output to and from the executable, `UCall` uses the filenames specified in the variables `$FileIn` and `$FileOut`, respectively. In most cases, the user will not need to change any of these global variables.

Examples for the evaluation of various cases of  $U_{4,5,6}$  integrals are shown in the file `example.m`, which is provided with the TVID program package.

### 3.3 Installation

TVID is available for download as a gzipped tar-ball. After saving it in the desired directory, it can be unpacked by the command

```
tar xzf tvid.tgz
```

After unpacking, the numerical C program must be compiled. A make file is provided for this purpose, which can be invoked with the simple command

```
make
```

The make file provided in version 1.0 has been tested on Scientific Linux 6. For other UNIX-type operating systems, the user may need to change the name of the C++ compiler (variable `CC`) or explicitly require the inclusion of certain standard libraries (variable `LIBS`) in the make file. The authors cannot guarantee that the installation process is successful on any operating system, but they appreciate any helpful suggestions, comments and bug reports.

## 4 Tests

The output of TVID have been compared to existing results in the literature for integrals with one and two independent mass scales [6–8]. At least ten digits agreement was found for all tested cases. A similar level of agreement was reached for a few comparisons of cases with maximal number of different mass scales with the work of Ref. [9].

The typical evaluation times of the numerical module (`ucall`) on a single core of an Intel Xeon CPU with 3.7 GHz are:  $\sim 0.1$  s for  $U_{4,\text{sub}}$ ;  $\lesssim 0.01$  s for  $U_{5,\text{sub},0}$ ; and 20–30 s for  $U_{6,\text{sub},0}$ .

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